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Maximum Hitting of a Set by Compressed Intersecting Families

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Abstract For a family $\mathcal{A}$ and a set Z, denote $\{A \in \mathcal{A}\colon A \cap Z \neq \emptyset\}$ by $\mathcal{A}(Z)$. For positive integers n and r, let $\mathcal{S}_{n,r}$ be the trivial *compressed intersecting family* $\{A \in \binom{[n]}{r}\colon 1 \in A\}$, where $[n] := \{1, \ldots, n\}$ and $\binom{[n]}{r} := \{A \subset [n]\colon |A| = r\}$. The following problem is considered: For $r \leq n/2$, which sets $Z \subseteq [n]$ have the property that $|\mathcal{A}(Z)| \leq |\mathcal{S}_{n,r}(Z)|$ for any compressed intersecting family $\mathcal{A} \subset \binom{[n]}{r}$? (The answer for the case $1 \in Z$ is given by the Erdős–Ko–Rado Theorem.) We give a complete answer for the case $|Z| \geq r$ and a partial answer for the much harder case $|Z| < r$. This paper is motivated by the observation that certain interesting results in extremal set theory can be proved by answering the question above for particular sets Z. Using our result for the special case when Z is the r-segment $\{2, \ldots, r+1\}$, we obtain new short proofs of two well-known Hilton–Milner theorems. At the other extreme end, by establishing that $|\mathcal{A}(Z)| \leq |\mathcal{S}_{n,r}(Z)|$ when Z is a final segment, we provide a new short proof of a Holroyd–Talbot extension of the Erdős-Ko-Rado Theorem.

Keywords Erdős–Ko–Rado Theorem · Intersecting family · Compressed family

Mathematics Subject Classification (2000) 05D05

1 Introduction

We start with some notation for sets. $\mathbb{N}$ is the set $\{1, 2, \ldots\}$ of all positive integers. For any $m, n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N}\colon m \leq i \leq n\}$ is denoted by $[m, n]$. For any integer n, $[n]$ denotes $[1, n]$ if $n \geq 1$, and $[n]$ is taken to be the empty set $\emptyset$

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if $n \le 0$. For a set X, the *power set* $\{A : A \subseteq X\}$ of X is denoted by 2^X. We denote $\{A \subseteq X : |A| = r\}$ by $\binom{X}{r}$, and $\{A \subseteq X : |A| \sim r\}$ by $\binom{X}{\sim r}$, where $\sim$ can be any of the relations $<, \le, \ge, >$.

We now develop some notation for families $\mathcal{F}$ of sets. Let $\mathcal{F}^{(r)} := \{F \in \mathcal{F} : |F| = r\}$ and $\mathcal{F}^{(\sim r)} := \{F \in \mathcal{F} : |F| \sim r\}$. Given a set X, let $\mathcal{F}(X) := \{F \in \mathcal{F} : F \cap X \neq \emptyset\}$ and $\mathcal{F}(\overline{X}) := \{F \in \mathcal{F} : F \cap X = \emptyset\}$. For $x \in X$, we abbreviate the notation $\mathcal{F}(\{x\})$ and $\mathcal{F}(\overline{\{x\}})$ to $\mathcal{F}(x)$ and $\mathcal{F}(\overline{x})$, respectively. Let $\mathcal{F}\langle x \rangle := \{F \backslash \{x\} : F \in \mathcal{F}(x)\}$.

A family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$. If $\mathcal{A} \subset 2^X$ and there exists $x \in X$ such that $x \in A$ for all $A \in \mathcal{A}$, then $\mathcal{A}$ is said to be *centred* with *centre* x; otherwise, $\mathcal{A}$ is said to be *non-centred*. For $a \in \bigcup_{A \in \mathcal{A}} A$, the centred sub-family $\mathcal{A}(a)$ of $\mathcal{A}$ is said to be a *star of* $\mathcal{A}$. Note that centred families are trivially intersecting.

Suppose $a_1 < \cdots < a_r, b_1 < \cdots < b_r, A := \{a_1, \ldots, a_r\}, B := \{b_1, \ldots, b_r\}$. If $a_i \le b_i$ for $i = 1, \ldots, r$, then we write $A \le B$, and if also $a_j < b_j$ for some $j \in [r]$, then we write $A < B$. A family $\mathcal{A}$ is said to be *compressed* if $A \in \mathcal{A}$ for any $A < B \in \mathcal{A}$.

Let $i, j \in [n]$. The well-known *compression operation* $\Delta_{i,j} : 2^{2^{[n]}} \to 2^{2^{[n]}}$ (see [3]) is defined by

$$\Delta_{i,j}(\mathcal{A}) := \{\delta_{i,j}(A) : A \in \mathcal{A}, \delta_{i,j}(A) \notin \mathcal{A}\} \cup \{A \in \mathcal{A} : \delta_{i,j}(A) \in \mathcal{A}\},$$

where $\delta_{i,j} : 2^{[n]} \to 2^{[n]}$ is defined by

$$\delta_{i,j}(A) = \begin{cases} (A \backslash \{j\}) \cup \{i\} & \text{if } i \notin A \text{ and } j \in A; \\ A & \text{otherwise.} \end{cases}$$

It is easy to see that a family $\mathcal{A} \subseteq 2^{[n]}$ is compressed if and only if $\Delta_{i,j}(\mathcal{A}) = \mathcal{A}$ for any $1 \le i < j \le n$. The paper [7] is an excellent survey on applications of the compression (also known as *shifting*) technique.

We denote the compressed star $\{A \in \binom{[n]}{r} : 1 \in A\}$ and the non-centred compressed intersecting family $\{A \in \binom{[n]}{r} : 1 \in A, A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$ by $\mathcal{S}_{n,r}$ and $\mathcal{N}_{n,r}$, respectively. We use the abbreviations $\mathcal{S}$ and $\mathcal{N}$ when n and r are clear from the context.

The following are two classical results in the literature on set combinatorics.

Theorem 1 (Erdős-Ko-Rado (EKR) Theorem [3]) *If $r \le n/2$ and $\mathcal{A}$ is an intersecting sub-family of $\binom{[n]}{r}$, then $|\mathcal{A}| \le |\mathcal{S}|$.*

Theorem 2 (Hilton–Milner Theorem [11]) *If $2 \le r \le n/2$ and $\mathcal{A}$ is a non-centred intersecting sub-family of $\binom{[n]}{r}$, then $|\mathcal{A}| \le |\mathcal{N}|$.*

The EKR Theorem inspired much research in extremal set theory that led to many beautiful results; the survey paper by Deza and Frankl [2] is recommended. Two short and beautiful proofs of this theorem are due to Daykin [1] and Katona [13]; Daykin's proof used a fundamental result known as the Kruskal–Katona Theorem [14,15], and Katona's proof introduced an elegant averaging technique referred to as the *cycle method*. In this paper, we will expand on ideas found in the original proof.

Theorem 2 is part of a more general result in [11], the proof of which is long and complicated. Many shorter and simpler proofs were obtained (see, for example [8,9]), and another one is given here.

In this paper, we pose the following question: Given $r \le n/2$, which sets $Z \subseteq [2, n]$ have the property—call it (*)—that $|\mathcal{A}(Z)| \le |\mathcal{S}(Z)|$ for any compressed intersecting family $\mathcal{A} \subset \binom{[n]}{r}$? As we shall now explain, if any of the conditions is violated, then the problem of maximising $|\mathcal{A}(Z)|$ is straightforward. If we allow $Z \subseteq [n]$ then the answer for the case $1 \in Z$ is given by Theorem 1; indeed, if $1 \in Z$ then $|\mathcal{A}(Z)| \le |\mathcal{S}(Z)|$ since $|\mathcal{A}(Z)| \le |\mathcal{A}| \le |\mathcal{S}| = |\mathcal{S}(Z)|$. If we allow $\mathcal{A}$ to be non-compressed, then we take $\mathcal{B}$ to be a star of $\binom{[n]}{r}$ whose centre is in Z and again apply Theorem 1 to obtain $|\mathcal{A}(Z)| \le |\mathcal{A}| \le |\mathcal{B}| = |\mathcal{B}(Z)|$. If we allow $\mathcal{A}$ to be non-intersecting, then obviously $|\mathcal{A}(Z)|$ is a maximum if $\mathcal{A} = \binom{[n]}{r}$. And if we allow $r > n/2$, then in the case $r > n/2$ we have that $\binom{[n]}{r}$ is intersecting and hence, since $\binom{[n]}{r}$ is compressed, we can again take $\mathcal{A} = \binom{[n]}{r}$.

The above question is motivated by the fact that, as is shown in Sect. 3, certain interesting results in extremal set theory such as Theorem 2 can be proved by obtaining an affirmative answer to the question for certain sets Z.

As the following examples demonstrate, not all non-empty sets $Z \subseteq [2, n]$ have property (*):

(i) $\emptyset \neq Z \subseteq [2, r+1], n \ge 2r$: If $\mathcal{A} = \mathcal{N}$ then $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$.

(ii) $\emptyset \neq Z \subseteq [2, 2r-1], |Z| \le r, n = 2r$: Let $\mathcal{A} := \binom{[2r-1]}{r} = \mathcal{S}(\overline{n}) \cup \mathcal{B}$, where $\mathcal{B} := \binom{[2,2r-1]}{r}$. So $|\mathcal{A}(Z)| - |\mathcal{S}(Z)| = |\mathcal{B}(Z)| - |\mathcal{S}(n)(Z)| = \left(\binom{2r-2}{r} - \binom{2r-2-|Z|}{r}\right) - \left(\binom{2r-2}{r-2} - \binom{2r-2-|Z|}{r-2}\right) = \binom{2r-2-|Z|}{r-2} - \binom{2r-2-|Z|}{r} > 0$.

(iii) $2r \in Z, \emptyset \neq Z \backslash \{2r\} \subseteq [2, r], n = 2r$: For $i = 1, \ldots, r$, let $A_i := [2, r] \cup \{r+i\}$ and $A_i' := [2r] \backslash A_i$. Let $\mathcal{A} := (\mathcal{S} \backslash \{A_1', \ldots, A_r'\}) \cup \{A_1, \ldots, A_r\}$. So $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$.

Note that, in each of the examples above, $\mathcal{A}$ is non-centred, compressed and intersecting.

By Theorem 2, if $Z \subseteq \binom{[2,n]}{>r}$ then $|\mathcal{A}(Z)| \le |\mathcal{A}| \le |\mathcal{N}| < |\mathcal{S}(Z)|$ for any non-centred intersecting family $\mathcal{A} \subset \binom{[n]}{r}$. So this settles the case $|Z| > r$; however, we will prove this directly and go on to obtain a new proof of Theorem 2. We will also settle the special case $|Z| = r$. The case $|Z| < r$ is far more challenging, and we will not determine fully which of these sets obey or disobey (*); however, some of them are captured by the following result.

Theorem 3 *Let $2 \le r \le n/2$, and let $\mathcal{A}$ be a compressed intersecting sub-family of $\binom{[n]}{r}$. Let $\emptyset \neq Z \subseteq [2, n]$ and $Y := Z \cap [2r]$. Suppose that at least one of the following holds:*

(a) $Y = \emptyset$;
(b) $|Z| \le r$ *and* $Y > W := [2r] \backslash ([2r - 2|Y|] \cup Y)$;
(c) $|Z| > r$.

Then $|\mathcal{A}(Z)| \le |\mathcal{S}(Z)|$.

Moreover, we can have $\mathcal{A}$ non-centred and $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ if and only if

(i) $r = 2$ *and either* $Z = Y \neq \{4\}$ *or* $\{2,3\} \subset Z \in \binom{[2,n]}{3}$*, or*
(ii) $2 < r = n/2$ *and* $Z \cap [2, r+1] \neq \emptyset$.

We point out that, if $|Z| = r = n/2$ and (b) does not hold, then (*) does not hold; see Theorem 5 below. It is perhaps surprising that the case $n = 2r$ is the tricky part of the proof of Theorem 3.

We now make two observations that should make the statement of the theorem easier to grasp:

– Suppose (b) holds. Let $U := [2r]\backslash[2r - 2|Y|]$. Clearly $W \subset U$. By definition of $<$ on members of $\binom{[n]}{r}$, we must have $Y \subset U$, because otherwise we get $|W| = 2r - (2r - 2|Y|) - |Y \cap U| = 2|Y| - |Y \cap U| > |Y|$, which means that Y and W are incomparable and hence contradicts $Y > W$. So $|W| = |Y| = |U|/2$, $W \cap Y = \emptyset$, and hence

$$Y \cup W = U. \tag{1}$$

– Clearly, if a compressed family is centred, then it must be a sub-family of $\mathcal{S}$. So $\mathcal{A}$ is non-centred if and only if $\mathcal{A} \not\subseteq \mathcal{S}$. We now determine the cases in which $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ holds for compressed proper sub-families $\mathcal{A}$ of $\mathcal{S}$. Let $m := \max\{z : z \in Z\}$ and $\mathcal{S}^* := \mathcal{S}\backslash\{A \in \mathcal{S} : A\backslash\{1\} \subset [n]\backslash[m]\}$. $\mathcal{S}^* \neq \mathcal{S}$ if and only if $m \leq n-r+1$. Also, $\mathcal{S}^*$ is compressed and $\mathcal{S}^*(Z) = \mathcal{S}(Z)$. If $m \leq n-r+1$, then take $A^* := \{1, m\} \cup ([n]\backslash[n-r+2])$, otherwise take $A^* := \{1\} \cup ([n]\backslash[n-r+1])$. So $A^* \in \mathcal{S}(Z)$ and $A \leq A^*$ for any $A \in \mathcal{S}^*$. Thus, if $\mathcal{A} \subset \mathcal{S}$ and $\mathcal{A}(Z) = \mathcal{S}(Z)$, then $A^* \in \mathcal{A}(Z)$ and hence $\mathcal{S}^* \subseteq \mathcal{A}$ (since $A^* \in \mathcal{S}^*$ and $\mathcal{A}$ is compressed).

If $x_1 < x_2 < \cdots < x_n$ and $m < n$, then we call the set $\{x_i : i \in [m+1, n]\}$ a *final* $(n-m)$*-segment of* $\{x_1, x_2, \ldots, x_n\}$. A consequence of Theorem 3 is as follows.

Corollary 1 *Let* $2 \leq r \leq n/2$*, and let* $\mathcal{A}$ *be a compressed intersecting sub-family of* $\binom{[n]}{r}$*. Let* Z *be a final segment of* $[n]$*. Then* $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$*.*

Moreover, if $\mathcal{A} \neq \mathcal{S}$*, then* $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ *if and only if* $n = 2r$*,* $|Z| \geq r$ *and* $|\mathcal{A}| = |\mathcal{S}|$*.*

Proof We have $Z = [m, n]$ for some $1 \leq m \leq n$. Let $Y := Z \cap [2r]$. If $Y = \emptyset$ or $|Z| > r$, then $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$ follows directly from Theorem 3. Suppose $|Z| \leq r$ and $Y \neq \emptyset$. Then $r+1 \leq m \leq 2r$ and $Y = [m, 2r]$. Let $W := [2r]\backslash([2r-2|Y|] \cup Y)$. So $W = [m - |Y|, m-1]$ and hence $W < Y$. So (b) of Theorem 3 holds and hence $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$.

We now prove the second part. Suppose $\mathcal{A} \neq \emptyset$. Since $\mathcal{A}$ is compressed, $[r] \in \mathcal{A} \cap \mathcal{S}$. If $n = 2r$ and $|Z| \geq r$, then all sets in $\binom{[n]}{r}\backslash\{[r]\}$ intersect Z, and hence $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ if we also have $|\mathcal{A}| = |\mathcal{S}|$. Conversely, suppose $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$. If $\mathcal{A} \subseteq \mathcal{S}$ then $\mathcal{A}(Z) = \mathcal{S}(Z)$ and hence, since $\mathcal{A}$ is compressed and $A < \{1\} \cup [n-r+2, n] \in \mathcal{S}(Z)$ for all $A \in \mathcal{S}$, we must actually have $\mathcal{A} = \mathcal{S}$. Suppose $\mathcal{A} \not\subseteq \mathcal{S}$. Then $\mathcal{A}$ is non-centred. It follows from (i) and (ii) of Theorem 3 that $n = 2r$ and $|Z| \geq r$. By Theorem 3, $|\mathcal{A}([2, n])| \leq |\mathcal{S}([2, n])|$ and hence $|\mathcal{A}| \leq |\mathcal{S}|$. So actually $|\mathcal{A}| = |\mathcal{S}|$ since $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$, $[r] \in \mathcal{A} \cap \mathcal{S}$ and all sets in $\binom{[2r]}{r}\backslash\{[r]\}$ intersect Z. □

In Sect. 3, we will show that the following extension of the EKR Theorem follows from the above corollary.

Theorem 4 (Holroyd, Talbot [12]) *Let* $X_1, \ldots, X_p$ *be distinct non-empty finite sets such that* $W := \bigcap_{k=1}^{p} X_k \neq \emptyset$, $X_i \cap X_j = W$ *for any* $i, j \in [p]$ *with* $i \neq j$, *and* $\mu := \min\{|X_i| : i \in [p]\} \geq 4$. *Let* $\mathcal{A}$ *be an intersecting sub-family of* $\mathcal{U} := \bigcup_{i=1}^{p} \binom{X_i}{r}$, *where* $2 \leq r \leq \mu/2$. *Then:*

(i) $|\mathcal{A}| \leq |\mathcal{U}(w)|$ *for any* $w \in W$;
(ii) *if* $r < \mu/2$ *and* $\mathcal{A}$ *is non-centred, then the inequality in* (i) *is strict.*

We mention that, in the literature, a family consisting of sets X_i as above is called a *sunflower* or *delta-system*. A classical result about sunflowers is the Erdős–Rado Theorem [4]. Sunflowers are used in the *kernel method* introduced in [10]; other applications are given in [5,6]. The maximal independent sets of the union of a *complete multipartite graph* and an *empty graph* form a sunflower; Holroyd and Talbot expressed Theorem 4 in these graph-theoretical terms.

The next theorem settles our problem for the special case $|Z| = r$.

Theorem 5 *Let* $Z \in \binom{[2,n]}{r}$, $2 \leq r \leq n/2$. *Let* $\mathcal{A}$ *be a compressed intersecting sub-family of* $\binom{[n]}{r}$ *such that* $\mathcal{A}(Z)$ *is of largest size. If*

(a) $\{2, 3\} \subseteq Z$ *and* $r \leq 3$, *or*
(b) $n = 2r$ *and* $[2r]\backslash Z \not< Z$, *or*
(c) $Z = [2, r+1]$,

then $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$, *otherwise* $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$.

In Sect. 3, we use the above result to obtain a short proof of both Theorem 2 and another Hilton–Milner result given by the following.

Theorem 6 (Hilton, Milner [11]) *Let* $r \leq n/2$, *and let* $\mathcal{A}$ *and* $\mathcal{B}$ *be non-empty sub-families of* $\binom{[n]}{r}$ *such that* $A \cap B \neq \emptyset$ *for any* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$. *Let* $\mathcal{A}_0 = \{A \in \binom{[n]}{r} : A \cap [r] \neq \emptyset\}$ *and* $\mathcal{B}_0 = \{[r]\}$. *Then* $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0| = \binom{n}{r} - \binom{n-r}{r} + 1$.

Two families $\mathcal{A}$ and $\mathcal{B}$ as in the above theorem are said to be *cross-intersecting*.

We now proceed to the proofs of Theorems 3 and 5.

2 Proofs of Main Results

We begin with a lemma concerning ordered pairs of sets in $\binom{[n]}{r}$.

Lemma 1 *Let* $A, B \in \binom{[n]}{r}$, *and let* $C \subseteq A \cap B$. *Then*

$$A < B \Leftrightarrow A\backslash C < B\backslash C.$$

Proof Suppose $A < B$. We must prove that $A\backslash C < B\backslash C$.

Suppose $C = \{c\}$. We have $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ for some $a_1 < \cdots < a_r$ and $b_1 < \cdots < b_r$. Since $A < B$, $c = a_p = b_q$ for some $p \geq q$. If

$p = q$ then the result is immediate. Suppose $p > q$. Then $A\backslash\{c\} = \{a_1^*, \ldots, a_{r-1}^*\}$ and $B\backslash\{c\} = \{b_1^*, \ldots, b_{r-1}^*\}$, where $a_i^* := a_i \le b_i =: b_i^*$ for $i = 1, \ldots, q-1$, $a_i^* := a_i \le b_i < b_{i+1} =: b_i^*$ for $i = q, \ldots, p-1$, and $a_i^* := a_{i+1} \le b_{i+1} =: b_i^*$ for $i = p, \ldots, r-1$. So $A\backslash C < B\backslash C$ as required.

The result for general C follows by a simple inductive argument.

Conversely, suppose $A\backslash C < B\backslash C$. Let $D := A\backslash C$, $E := B\backslash C$. We have $D < E$ and must prove that $D \cup C < E \cup C$.

Suppose $C = \{c\}$. We have $D = \{d_1, \ldots, d_s\}$ and $E = \{e_1, \ldots, e_s\}$ for some $d_1 < \cdots < d_s$ and $e_1 < \cdots < e_s$. If $c < d_1$ or $c > e_s$, then the result is immediate; so we may assume that $c \in [d_1+1, e_s-1]$. Let $j := \max\{i \in [s] \colon d_i < c\}$, $k := \min\{i \in [s] \colon c < e_i\}$. Then $D \cup \{c\} = \{d_1^*, \ldots, d_{s+1}^*\}$ and $E \cup \{c\} = \{e_1^*, \ldots, e_{s+1}^*\}$, where $d_i^* := d_i$ for $i = 1, \ldots, j$, $d_{j+1}^* := c$, $d_i^* := d_{i-1}$ for $i = j+2, \ldots, s+1$, $e_i^* := e_i$ for $i = 1, \ldots, k-1$, $e_k^* := c$, $e_i^* := e_{i-1}$ for $i = k+1, \ldots, s+1$. Note that $k \le j+1$ since $D < E$. If $k = j+1$ then clearly $d_i^* \le e_i^*$, $i = 1, \ldots, s+1$, with at least one strict inequality. If $k < j+1$ then $d_i^* = d_i \le e_i = e_i^*$ for $i = 1, \ldots, k-1$, $d_i^* = d_i < c = e_k^* \le e_i^*$ for $i = k, \ldots, j$, $d_{j+1}^* = c < e_{j+1}^*$, and $d_i^* = d_{i-1} \le e_{i-1} = e_i^*$ for $i = j+2, \ldots, s+1$. So $D \cup C < E \cup C$ as required.

The result for general C again follows by a simple inductive argument. □

Lemma 2 *If $\mathcal{A}$ is a compressed sub-family of $2^{[n]}$ and $Z, \{a, b\} \subset [n]$, $a < b$, then $|\mathcal{A}(Z)| \le |\mathcal{A}(\delta_{a,b}(Z))|$.*

Proof Suppose $Z' := \delta_{a,b}(Z) \neq Z$. Taking $Z'' := Z \cap Z'$, we then have $Z = Z'' \cup \{b\} \neq Z''$ and $Z' = Z'' \cup \{a\} \neq Z''$. Since $\mathcal{A}$ is compressed, $\Delta_{a,b}(\mathcal{A}(\overline{Z''})(b)(\overline{a})) \subseteq \mathcal{A}(\overline{Z''})(a)(\overline{b})$. So $|\mathcal{A}(\overline{Z''})(a)(\overline{b})| \ge |\mathcal{A}(\overline{Z''})(b)(\overline{a})|$. We therefore have

$$\begin{aligned}|\mathcal{A}(Z')| - |\mathcal{A}(Z)| &= (|\mathcal{A}(Z'')| + |\mathcal{A}(\overline{Z''})(a)|) - (|\mathcal{A}(Z'')| + |\mathcal{A}(\overline{Z''})(b)|)\\ &= (|\mathcal{A}(\overline{Z''})(a)(b)| + |\mathcal{A}(\overline{Z''})(a)(\overline{b})|) - (|\mathcal{A}(\overline{Z''})(b)(a)| + |\mathcal{A}(\overline{Z''})(b)(\overline{a})|) \ge 0,\end{aligned}$$

and hence the result. □

Proof of Theorem 3 It is easy to check the result for $r = 2$ because $\mathcal{N}_{n,2} = \binom{[3]}{2}$ is the only non-centred compressed intersecting sub-family of $\binom{[n]}{2}$.

We now consider $r \ge 3$. We shall assume that

$$|\mathcal{A}'(Z)| \le |\mathcal{A}(Z)| \quad \text{for any compressed intersecting } \mathcal{A}' \subset \binom{[n]}{r}. \tag{2}$$

We need to prove two things:

(I) $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ and, if $\mathcal{A}$ is non-centred, then (ii) holds;
(II) if (ii) holds then there exists a non-centred compressed intersecting family $\mathcal{A}' \subset \binom{[n]}{r}$ such that $|\mathcal{A}'(Z)| = |\mathcal{S}(Z)|$.

Let us quickly verify (II). So suppose $n = 2r$ (which gives $Y = Z$) and $Z \cap [2, r+1] \neq \emptyset$. Let $A_1 := [2, r+1]$ and $A_2 := \{1\} \cup [r+2, \ldots, 2r]$. We have $\mathcal{N} = (\mathcal{S}\backslash\{A_2\}) \cup \{A_1\}$ and hence $|\mathcal{N}| = |\mathcal{S}|$. If $|Z| > r$ then $|\mathcal{N}(Z)| = |\mathcal{N}| = |\mathcal{S}| = |\mathcal{S}(Z)|$ trivially. Suppose $|Z| \le r$. Then (b) holds, and this implies that $2r \in Z$ (see (1)) and hence $A_2 \cap Z \neq \emptyset$. Now we are given that $Z \cap A_1 \neq \emptyset$. So $|\mathcal{N}(Z)| = |\mathcal{S}(Z)|$. Hence (II).

We now prove (I) using induction on n.

Case 1: $n = 2r$. So $Y = Z \neq \emptyset$.

Sub-case 1.1: $|Z| > r$. Having $\mathcal{A}$ intersecting means that $[2r]\backslash A \notin \mathcal{A}$ for all $A \in \mathcal{A}$, and hence $|\mathcal{A}| \leq \frac{1}{2}\binom{2r}{r} = \binom{2r-1}{r-1} = |\mathcal{S}|$. So this case is straightforward since here $\mathcal{A}(Z) = \mathcal{A}$ and $\mathcal{S}(Z) = \mathcal{S}$.

Sub-case 1.2: $|Z| = r$. So $Z \cap [2, r+1] \neq \emptyset$. Suppose $Z \in \mathcal{A}$. Then, since $[2r]\backslash Z = W < Z$ (by (b) and (1)) and $\mathcal{A}$ is compressed, we have $[2r]\backslash Z \in \mathcal{A}$; but this is a contradiction as $\mathcal{A}$ is intersecting. So $Z \notin \mathcal{A}$. Thus, since Z and $[2r]\backslash Z$ are both members of $\binom{[2r]}{r}$ that are not in $\mathcal{A}(Z)$, we have $|\mathcal{A}(Z)| \leq \frac{1}{2}\left(\binom{2r}{r} - 2\right) = |\mathcal{S}(Z)|$ (note that (b) $\Rightarrow Z > [2r]\backslash Z \Rightarrow 1 \in [2r]\backslash Z \Rightarrow \{[2r]\backslash Z\} = \mathcal{S}(\overline{Z})$), because having $\mathcal{A}(Z)$ intersecting means that, for all $A \in \binom{[2r]}{r}$, at most one of A and $[2r]\backslash A$ is in $\mathcal{A}(Z)$. By (2), $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$.

Sub-case 1.3: $|Z| < r$. Let $\mathcal{A}_1 \cup \mathcal{A}_2$ be the partition of $\mathcal{A}(Z)$ with $\mathcal{A}_1 := \{A \in \mathcal{A}(Z) \colon Z\backslash A \neq \emptyset\}$ and $\mathcal{A}_2 := \{A \in \mathcal{A}(Z) \colon Z \subset A\}$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be defined similarly. Let $f \colon \mathcal{A}_1 \to \mathcal{S}_1$ be defined by $f(A) := A$ if $1 \in A$, and $f(A) := [2r]\backslash A$ if $1 \notin A$ ($A \in \mathcal{A}_1$). So f is injective, because if $1 \in C \in \mathcal{A}_1$, $1 \notin D \in \mathcal{A}_1$, and $f(D) = f(C)$, then $[2r]\backslash D = C \in \mathcal{A}$, contradicting the assumption that $\mathcal{A}$ is intersecting. Therefore $|\mathcal{A}_1| \leq |\mathcal{S}_1|$.

Now consider $\mathcal{A}_2$, and suppose there exist $C, D \in \mathcal{A}_2$ such that $(C \cap D)\backslash Z = \emptyset$. Thus, taking $E := [2r]\backslash D$ and $F := E\backslash C$, we have $C\backslash E = Z$ and $F \subset [2r]\backslash Z$. Note that $|F| = |([2r]\backslash D)\backslash C| = |[2r]\backslash(C \cup D)| = 2r - (|C| + |D| - |C \cap D|) = |Z|$. Since $Y = Z$, we have $W\backslash F \subset W \subset [2r-2|Z|+1, 2r]$ by (b), and $F\backslash W \subset [2r]\backslash(Z \cup W) = [2r-2|Z|]$ by (1). So $F\backslash W \leq W\backslash F$. By Lemma 1, $F \leq W$. So we have $E\backslash C \leq W < Z = C\backslash E$, and hence Lemma 1 gives us $E < C$. Since $\mathcal{A}$ is compressed and $C \in \mathcal{A}$, we get $E \in \mathcal{A}$, which is a contradiction because $E \cap D = \emptyset$ and $D \in \mathcal{A}$. Therefore

$$(C \cap D)\backslash Z \neq \emptyset \quad \text{for all } C, D \in \mathcal{A}_2. \tag{3}$$

Next, define $\mathcal{X} := \{A\backslash Z \colon A \in \mathcal{A}_2\}$. Let $n' := 2r - |Z|$ and $r' := r - |Z|$. Let $x_1, \ldots, x_{n'}$ be the distinct elements of $[2r]\backslash Z$ listed in increasing order. Let $g : [2r]\backslash Z \to [n']$ such that $g(x_i) := i$, and let $h : \binom{[2r]\backslash Z}{r'} \to \binom{[n']}{r'}$ such that, if A is a set $\{a_1, \ldots, a_{r'}\}$ in $\binom{[2r]\backslash Z}{r'}$, then $h(A) := \{g(a_1), \ldots, g(a_{r'})\}$. So g and h are bijections. Let $\mathcal{X}' := \{h(A) \colon A \in \mathcal{X}\}$. By (3), $\mathcal{X}$ is intersecting and hence $\mathcal{X}'$ is intersecting. $\mathcal{X}'$ is also compressed because

$$\begin{aligned} A < B \in \mathcal{X}' &\Rightarrow C := h^{-1}(A) < D := h^{-1}(B) \in \mathcal{X} \\ &\Rightarrow (C \cup Z)\backslash Z < (D \cup Z)\backslash Z \in \mathcal{X} \\ &\Rightarrow C \cup Z < D \cup Z \in \mathcal{A}_2 \quad \text{(by Lemma 1)} \\ &\Rightarrow C \cup Z \in \mathcal{A}_2 \quad \text{(since } \mathcal{A} \text{ is compressed)} \\ &\Rightarrow C \in \mathcal{X} \Rightarrow A \in \mathcal{X}'. \end{aligned}$$

Let $\mathcal{Y} := \{A \backslash Z \colon A \in \mathcal{S}_2\}$ and $\mathcal{Y}' := \{h(A) \colon A \in \mathcal{Y}\} = \mathcal{S}_{n',r'}$. Now $\mathcal{X}' \subseteq \binom{[n']}{r'}$, where $1 \leq r' < n'/2$ as $|Z| < r = n/2$. If $r' = 1$ then we trivially have $\mathcal{X} \subseteq \mathcal{Y}$. If $r' > 1$ then we set $Z' := [2, n']$ and, since $\mathcal{X}' = \mathcal{X}'(Z')$ and $\mathcal{Y}' = \mathcal{Y}'(Z')$, we apply the inductive hypothesis to obtain that $|\mathcal{X}'| \leq |\mathcal{Y}'|$, and that equality holds only if $\mathcal{X}' = \mathcal{Y}'$. So $|\mathcal{A}_2| \leq |\mathcal{S}_2|$, and equality holds only if $\mathcal{A}_2 = \mathcal{S}_2$. Since $|\mathcal{A}_1| \leq |\mathcal{S}_1|$, $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$. By (2), $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$. So $|\mathcal{A}_1| = |\mathcal{S}_1|$, $|\mathcal{A}_2| = |\mathcal{S}_2|$, and hence $\mathcal{A}_2 = \mathcal{S}_2$.

Suppose $Z \cap [2, r+1] = \emptyset$. So $Z \subseteq [r+2, 2r]$ and hence, since $\mathcal{A}_2 = \mathcal{S}_2$, $A^* := \{1\} \cup [r+2, 2r]$ is in $\mathcal{A}_2$. Since $\mathcal{A}$ is compressed and $A \leq A^*$ for all $A \in \mathcal{S}$, we obtain $\mathcal{S} \subseteq \mathcal{A}$. Together with $|\mathcal{A}| \leq |\mathcal{S}|$ (see Sub-case 1.1), this gives us $\mathcal{A} = \mathcal{S}$.

Case 2: $n > 2r$. Let $n' := n - 1$, $r' := r - 1$. We have $\mathcal{A}(\overline{n}), \mathcal{S}(\overline{n}) \subset \binom{[n']}{r}$ and $\mathcal{A}\langle n \rangle, \mathcal{S}\langle n \rangle \subset \binom{[n']}{r'}$. Note that $r \leq n'/2$ and $r' < n'/2$ (as we now have $r < n/2$). Also note that $\mathcal{A}\langle n \rangle$ and $\mathcal{A}(\overline{n})$ are compressed. We now show that $\mathcal{A}\langle n \rangle \cup \mathcal{A}(\overline{n})$ is intersecting.

Suppose $A \cap B \cap [n'] = \emptyset$ for some $A, B \in \mathcal{A}$. So $A \cap B = \{n\}$ (as $\mathcal{A}$ is intersecting). Since $|A \cup B| \leq 2r - 1 < n'$, $[n] \backslash (A \cup B) \neq \emptyset$. Let $a \in [n] \backslash (A \cup B)$. Since $A' := (A \backslash \{n\}) \cup \{a\} < A$ and $\mathcal{A}$ is compressed, $A' \in \mathcal{A}$. But $A' \cap B = \emptyset$, a contradiction. So $A \cap B \cap [n'] \neq \emptyset$ for any $A, B \in \mathcal{A}$, and hence $\mathcal{A}\langle n \rangle \cup \mathcal{A}(\overline{n})$ is intersecting as required.

Sub-case 2.1: $n \notin Z$. It is immediate from the inductive hypothesis that $|\mathcal{A}(\overline{n})(Z)| \leq |\mathcal{S}(\overline{n})(Z)|$ and $|\mathcal{A}\langle n \rangle(Z)| \leq |\mathcal{S}\langle n \rangle(Z)|$. Since $|\mathcal{A}(Z)| = |\mathcal{A}(\overline{n})(Z)| + |\mathcal{A}\langle n \rangle(Z)|$, it follows that $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$. By (2), we actually have $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$, giving $|\mathcal{A}(\overline{n})(Z)| = |\mathcal{S}(\overline{n})(Z)|$ and $|\mathcal{A}\langle n \rangle(Z)| = |\mathcal{S}\langle n \rangle(Z)|$. It remains to show that $\mathcal{A}$ is centred.

Consider $|\mathcal{A}\langle n \rangle(Z)| = |\mathcal{S}\langle n \rangle(Z)|$. Since $r' < n'/2$, it follows by the inductive hypothesis that $\mathcal{A}\langle n \rangle$ is centred (note that conditions (a)–(c) imply that (i) cannot hold for $\mathcal{A}\langle n \rangle(Z)$). So $\mathcal{A}\langle n \rangle(Z) = \mathcal{S}\langle n \rangle(Z)$ as $\mathcal{A}\langle n \rangle$ is compressed. Hence $\mathcal{A}(n)(Z) = \mathcal{S}(n)(Z)$.

Let $m := \max\{z \colon z \in Z\}$. If $r > 3$ then we take F_1 to be the final $(r-3)$-segment for $[n] \backslash \{1, m, n\}$; otherwise, we take F_1 to be $\emptyset$. Let $S_1 := \{1, m, n\} \cup F_1$ (recall that we are dealing with $r \geq 3$). Since $Z \subseteq [2, n]$, if $|Z| \geq r+1$ then $m \geq r+2$. Suppose $|Z| \leq r$. If $Y = \emptyset$ then $m > 2r$, and if $Y \neq \emptyset$ then, by (b) and (1), we have $2r \in Y$, and hence $m \geq 2r$. So we have $m \geq r+2$. Suppose that $\mathcal{A}$ is non-centred. Given that $\mathcal{A}$ is compressed, we then have $[2, r+1] \in \mathcal{A}$, which is a contradiction because $[2, r+1] \cap S_1 = \emptyset$, $S_1 \in \mathcal{S}(n)(Z) = \mathcal{A}(n)(Z)$ and $\mathcal{A}$ is intersecting. So $\mathcal{A}$ is centred.

Sub-case 2.2: $n \in Z$. Suppose $Z \neq [2, n]$. Let $m' := \max\{a \colon a \in [n] \backslash Z\}$ and $Z' := \delta_{m',n}(Z)$. So $n \notin Z'$. It is easy to check that Z' also satisfies one of (a), (b) and (c). Therefore, as in Sub-case 2.1, we have $|\mathcal{A}(Z')| \leq |\mathcal{S}(Z')|$, and equality holds only if $\mathcal{A}$ is centred. Now $|\mathcal{S}(Z)| = |\mathcal{S}(Z')|$ and, by Lemma 2, $|\mathcal{A}(Z)| \leq |\mathcal{A}(Z')|$. So $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$. By (2), $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$. So $|\mathcal{A}(Z')| = |\mathcal{S}(Z')|$ and hence $\mathcal{A}$ is centred.

Now suppose $Z = [2, n]$. Then, by setting $Z'' := Z \backslash \{n\}$ and applying the inductive hypothesis, we get $|\mathcal{A}(\overline{n})| = |\mathcal{A}(\overline{n})(Z'')| \leq |\mathcal{S}(\overline{n})(Z'')| = |\mathcal{S}(\overline{n})|$ and $|\mathcal{A}\langle n \rangle| =$

$|\mathcal{A}\langle n\rangle(Z'')| \leq |\mathcal{S}\langle n\rangle(Z'')| = |\mathcal{S}\langle n\rangle|$, and (since $r' < n'/2$) the latter inequality is an equality only if $\mathcal{A}\langle n\rangle(Z'')$ is centred and hence $\mathcal{A}\langle n\rangle = \mathcal{S}\langle n\rangle$ (as $\mathcal{A}$ is compressed and $\mathcal{A}\langle n\rangle = \mathcal{A}\langle n\rangle(Z'')$). Now we have $|\mathcal{A}(Z)| = |\mathcal{A}(\overline{n})| + |\mathcal{A}\langle n\rangle| \leq |\mathcal{S}(\overline{n})| + |\mathcal{S}\langle n\rangle| = |\mathcal{S}| = |\mathcal{S}(Z)|$. By (2), $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$. So $|\mathcal{A}\langle n\rangle| = |\mathcal{S}\langle n\rangle|$ and hence $\mathcal{A}\langle n\rangle = \mathcal{S}\langle n\rangle$. Since $r' < n'/2$, for any $A \in \binom{[2,n']}{r}$ there exists $B \in \mathcal{S}\langle n\rangle$ such that $A \cap B = \emptyset$. Since $\mathcal{A}\langle n\rangle \cup \mathcal{A}(\overline{n})$ is intersecting and $\mathcal{A}\langle n\rangle = \mathcal{S}\langle n\rangle$, it follows that $\mathcal{A}(\overline{n}) \subset \mathcal{S}$ and hence $\mathcal{A} \subseteq \mathcal{S}$. □

We now come to the proof of Theorem 5, for which we need the following second lemma concerning ordered pairs of sets in $\binom{[n]}{r}$.

Lemma 3 *Let X be a finite subset of $\mathbb{N}$, and let $A, B \in \binom{X}{r}$. Then*

$$A < B \Leftrightarrow X\backslash B < X\backslash A.$$

Proof (⇒) We use induction on $|X|$ to prove that $X\backslash B < X\backslash A$ if $A < B$. The case $|X| \leq 2$ is trivial. Consider $|X| > 2$. Suppose $A < B$. Let $C := A \cap B$, $Y := X\backslash C$.

Suppose $C \neq \emptyset$. Let $D := A\backslash C$, $E := B\backslash C$. So $D, E \in \binom{Y}{r-|C|}$. By Lemma 1, $D < E$. By the inductive hypothesis, $Y\backslash E < Y\backslash D$. The result follows since $Y\backslash E = X\backslash B$ and $Y\backslash D = X\backslash A$.

Now suppose $C = \emptyset$. So $|X| \geq |A|+|B| = 2r$. If $|X| = 2r$ then $X\backslash B = A < B = X\backslash A$. Suppose $|X| > 2r$. Let $c \in X\backslash(A\cup B)$. Let $Y := X\backslash\{c\}$, $H := Y\backslash B$, $I := Y\backslash A$. By the inductive hypothesis, $H < I$. By Lemma 1, $X\backslash B = H\cup\{c\} < I\cup\{c\} = X\backslash A$.

(⇐) If $X\backslash B < X\backslash A$ then, by the above, $X\backslash(X\backslash A) < X\backslash(X\backslash B)$ and hence $A < B$. □

Proof of Theorem 5 For the same reason specified in the proof of Theorem 3, the case $r = 2$ is straightforward. So we consider $r \geq 3$.

We start by demonstrating the lower bound $|\mathcal{S}(Z)| + 1 \leq |\mathcal{A}(Z)|$ for each of the cases (a), (b), (c). For case (a) (where $r = 3$), take $\mathcal{A}_{(a)} := \{A \in \binom{[n]}{3} \colon |A \cap [3]| \geq 2\}$. For case (b), take $\mathcal{A}_{(b)}$ to be the union of $\mathcal{A}'_{(b)} := \{A \in \binom{[2r]}{r} \colon A \leq Z\}$ and $\mathcal{A}''_{(b)} := \mathcal{S}\backslash\mathcal{B}$, where $\mathcal{B} = \{B \in \mathcal{S} \colon [2r]\backslash B \in \mathcal{A}'_{(b)}\}$. For case (c), take $\mathcal{A}_{(c)} := \mathcal{N}$. It is straightforward that $\mathcal{A}_{(a)}$ and $\mathcal{A}_{(c)}$ are compressed and intersecting, and that they attain the required lower bound. We now prove the less straightforward fact that the same holds for $\mathcal{A}_{(b)}$.

By definition of $\mathcal{A}'_{(b)}$, if $A < B \in \mathcal{A}'_{(b)}$ then $A < B \leq Z$ and hence $A \in \mathcal{A}'_{(b)}$; so $\mathcal{A}'_{(b)}$ is compressed. Now suppose $A < B \in \mathcal{A}''_{(b)}$. Then, by Lemma 3 and the definition of $\mathcal{A}''_{(b)}$, we have $[2r]\backslash A > [2r]\backslash B \notin \mathcal{A}'_{(b)}$ and hence, since $\mathcal{A}'_{(b)}$ is compressed, $[2r]\backslash A \notin \mathcal{A}'_{(b)}$. Also, $A \in \mathcal{S}$ since $A < B \in \mathcal{A}''_{(b)} \subset \mathcal{S}$. So $A \in \mathcal{A}''_{(b)}$, which proves that $\mathcal{A}''_{(b)}$ is compressed. Thus, as required, $\mathcal{A}_{(b)}$ is compressed because clearly, in general, the union of two compressed families is compressed.

Suppose $A, B \in \mathcal{A}_{(b)}$. It is straightforward that if $A \in \mathcal{A}''_{(b)}$ or $B \in \mathcal{A}''_{(b)}$, then $A \cap B \neq \emptyset$. Now suppose $A, B \in \mathcal{A}'_{(b)}$ and $A \cap B = \emptyset$. Then $A \leq Z$, $B \leq Z$ and $A = [2r]\backslash B$. Applying Lemma 3, we get $[2r]\backslash Z \leq [2r]\backslash B = A \leq Z$, a contradiction to (b). So $\mathcal{A}'_{(b)}$ is intersecting and hence $\mathcal{A}_{(b)}$ is intersecting.

For any $B \in \mathcal{B}$, the complement $[2r]\backslash B$ of B is in $\mathcal{A}'_{(b)}\backslash\mathcal{S}$. So $|\mathcal{B}| \leq |\mathcal{A}'_{(b)}\backslash\mathcal{S}|$. Let $A \in \mathcal{A}'_{(b)}\backslash\mathcal{S}$. So $1 \notin A$, the complement $A' := [2r]\backslash A$ of A is in $\mathcal{S}$, and hence

$A' \in \mathcal{B}$ since $[2r]\backslash A' = A \in \mathcal{A}'_{(b)}$. So $|\mathcal{A}'_{(b)}\backslash\mathcal{S}| \leq |\mathcal{B}|$. Together with $|\mathcal{B}| \leq |\mathcal{A}'_{(b)}\backslash\mathcal{S}|$, this gives us $|\mathcal{B}| = |\mathcal{A}'_{(b)}\backslash\mathcal{S}|$. Now $|\mathcal{A}_{(b)}| = |\mathcal{A}''_{(b)}| + |\mathcal{A}'_{(b)}\backslash\mathcal{A}''_{(b)}| \geq |\mathcal{S}| - |\mathcal{B}| + |\mathcal{A}'_{(b)}\backslash\mathcal{S}| = |\mathcal{S}|$. Since $\mathcal{A}_{(b)}$ is intersecting, $[2r]\backslash A \in \binom{[2r]}{r}\backslash\mathcal{A}_{(b)}$ for any $A \in \mathcal{A}_{(b)}$. So $|\mathcal{A}_{(b)}| \leq \frac{1}{2}\binom{2r}{r} = |\mathcal{S}|$. Together with $|\mathcal{A}_{(b)}| \geq |\mathcal{S}|$, this gives us $|\mathcal{A}_{(b)}| = |\mathcal{S}|$. Now $Z' := [2r]\backslash Z$ is the unique set in $\binom{[2r]}{r}$ that does not intersect Z, $Z' \in \mathcal{S}$ (as $Z \in \binom{[2,n]}{r}$), and $Z' \notin \mathcal{A}_{(b)}$ (as $Z \in \mathcal{A}_{(b)}$ and $\mathcal{A}_{(b)}$ is intersecting). So $|\mathcal{S}(Z)| = |\mathcal{S}|-1$ and $|\mathcal{A}_{(b)}(Z)| = |\mathcal{A}_{(b)}|$. Since $|\mathcal{A}_{(b)}| = |\mathcal{S}|$, $|\mathcal{A}_{(b)}(Z)| = |\mathcal{S}(Z)| + 1$ as required.

The result now follows if we prove the upper bound $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)| + 1$ and that equality holds only if one of (a), (b) and (c) holds.

Case 1: $n = 2r$. It is immediate that therefore $|\mathcal{A}(Z)| \leq |\mathcal{S}| = |\mathcal{S}(Z)| + 1$ because $|\mathcal{A}| \leq \frac{1}{2}\binom{2r}{r} = |\mathcal{S}|$ (see proof of Theorem 3). Suppose (b) does not hold, i.e. $[n]\backslash Z < Z$. Then, by Theorem 3, $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$. So $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$ only if (b) holds.

Case 2: $n > 2r$. As in the proof of Theorem 3, $\mathcal{A}(\overline{n}) \subset \binom{[n']}{r}$ ($n' = n - 1$) and $\mathcal{A}\langle n\rangle \subset \binom{[n']}{r'}$ ($r' = r - 1$) are compressed and intersecting, and $\mathcal{A}\langle n\rangle \cup \mathcal{A}(\overline{n})$ is intersecting. Also recall that $r \leq n'/2$ and $r' < n'/2$.

Sub-case 2.1: $n \notin Z$. So $Z \in \binom{[2,n']}{r}$. By the inductive hypothesis, we have $|\mathcal{A}(\overline{n})(Z)| \leq |\mathcal{S}(\overline{n})(Z)| + 1$. By Theorem 3, $|\mathcal{A}\langle n\rangle(Z)| \leq |\mathcal{S}\langle n\rangle(Z)|$ (as $|Z| > r'$), and equality holds only if either $\mathcal{A}\langle n\rangle$ is centred or $r' = 2$ and $\{2, 3\} \subset Z$ (because $r' < n'/2$ and, if $r' = 2$ and $Z = Y := Z \cap [2r']$, then $Z = \{2, 3, 4\}$ since $|Z| = r' + 1$). So $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)| + 1$, and equality holds only if $\mathcal{A}\langle n\rangle \subseteq \mathcal{S}\langle n\rangle$ (as $\mathcal{A}\langle n\rangle$ is compressed) or (a) holds. Suppose $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|+1$ and (a) is not the case. So $|\mathcal{A}(\overline{n})(Z)| = |\mathcal{S}(\overline{n})(Z)| + 1$, $|\mathcal{A}\langle n\rangle(Z)| = |\mathcal{S}\langle n\rangle(Z)|$ and $\mathcal{A}\langle n\rangle \subseteq \mathcal{S}\langle n\rangle$. The last two relations yield $\mathcal{A}\langle n\rangle(Z) = \mathcal{S}\langle n\rangle(Z)$, and the first relation yields $\mathcal{A}(\overline{n}) \nsubseteq \mathcal{S}(\overline{n})$, implying that $A^* := [2, r + 1] \in \mathcal{A}(\overline{n})$ as $\mathcal{A}(\overline{n})$ is compressed. Suppose $Z \neq A^*$. Then, since $\mathcal{A}\langle n\rangle(Z) = \mathcal{S}\langle n\rangle(Z)$, we can choose a set A' in $\mathcal{A}\langle n\rangle(Z)$ that does not intersect A^*, but this is a contradiction because $\mathcal{A}\langle n\rangle\cup\mathcal{A}(\overline{n})$ is intersecting. So $Z = A^*$, i.e. (c) holds.

Sub-case 2.2: $n \in Z$. Let $m := \max\{a : a \in [n]\backslash Z\}$ and $Z' := \delta_{m,n}(Z)$. So $n \notin Z'$ and clearly Z' does not satisfy (c). Thus, as we have shown in Sub-case 2.1, we get $|\mathcal{A}(Z')| \leq |\mathcal{S}(Z')|+1$, and equality holds only if $\{2, 3\} \subset Z'$ and $r \leq 3$, in which case Z and r satisfy (a). The result follows since $|\mathcal{S}(Z)| = |\mathcal{S}(Z')|$ and $|\mathcal{A}(Z)| \leq |\mathcal{A}(Z')|$ by Lemma 2. □

3 Theorems 2, 4, 6 from Theorems 3, 5

Two fundamental properties of a compression $\Delta_{i,j}$ ($i, j \in [n]$) on a family $\mathcal{A} \subseteq 2^{[n]}$ are that $|\Delta_{i,j}(\mathcal{A})| = |\mathcal{A}|$ and that, if $\mathcal{A}$ is intersecting, then $\Delta_{i,j}(\mathcal{A})$ is intersecting; both are easy to check. Call $\Delta_{i,j}$ a *left-compression* if $i < j$. Call $\Delta_{i,j}$ a *proper compression on* $\mathcal{A}$ if $\Delta_{i,j}(\mathcal{A}) \neq \mathcal{A}$. It only takes a finite number of proper left-compressions for a family $\mathcal{A} \subset 2^{[n]}$ to become invariant under any left-compression (as the positive quantity $\sum_{A\in\mathcal{A}}\sum_{i\in A} i$ gets smaller after each proper left-compression), in which case $\mathcal{A}$ is clearly compressed.

We now prove Theorems 6, 2 and 4 in the given order. Theorem 6 will be used for proving Theorem 2.

Proof of Theorem 6 We may start by assuming that $[r] \in \mathcal{A}$. We now show that we may also assume that $[r] \in \mathcal{B}$. Indeed, suppose $[r] \notin \mathcal{B}$. Let $\mathcal{A}' := \{A \cup \{n+1\} \colon A \in \mathcal{A}\}$ and $\mathcal{B}' := \{B \cup \{n+2\} \colon B \in \mathcal{B}\}$. Given that $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting sub-families of $\binom{[n]}{r}$, $\mathcal{C} := \mathcal{A}' \cup \mathcal{B}'$ is an intersecting sub-family of $\binom{[n+2]}{r+1}$. Let B be a set $\{b_1, \ldots, b_r\}$ in $\mathcal{B}$; so $B' := B \cup \{n+2\}$ is in $\mathcal{B}'$. Let $\mathcal{D}$ be the family obtained by applying the left-compressions $\Delta_{1,b_1}, \ldots, \Delta_{r,b_r}$ to $\mathcal{C}$. This clearly gives us $[r] \cup \{n+1\}, [r] \cup \{n+2\} \in \mathcal{D}$. By setting $\mathcal{E} := \{D \backslash \{n+1\} \colon D \in \mathcal{D}\}$ and $\mathcal{F} := \{D \backslash \{n+2\} \colon D \in \mathcal{D}\}$, we then obtain $[r] \in \mathcal{E} \cap \mathcal{F}$. Also note that $\mathcal{E}, \mathcal{F} \subset \binom{[n]}{r}$, $|\mathcal{A}| = |\mathcal{E}|, |\mathcal{B}| = |\mathcal{F}|$. Since $\mathcal{C}$ is intersecting, $\mathcal{D}$ is intersecting. Clearly we therefore have that $\mathcal{E}$ and $\mathcal{F}$ are cross-intersecting. We may therefore assume that $\mathcal{A} = \mathcal{E}$ and $\mathcal{B} = \mathcal{F}$. So $[r] \in \mathcal{A} \cap \mathcal{B}$.

Let $f \colon \binom{[n]}{r} \to \binom{[3,n+2]}{r}$ such that, if A is a set $\{a_1, \ldots, a_r\}$ in $\binom{[n]}{r}$, then $f(A) := \{a_1+2, \ldots, a_r+2\}$. Let $\mathcal{G} := \{\{1\} \cup f(A) \colon A \in \mathcal{A}\}$, $\mathcal{H} := \{\{2\} \cup f(B) \colon B \in \mathcal{B}\}$, $\mathcal{I} := \{A \in \binom{[n+2]}{r+1} : [2] \subset A\}$. Let $\mathcal{J}$ be the disjoint union $\mathcal{G} \cup \mathcal{H} \cup \mathcal{I}$. So $\mathcal{J}$ is an intersecting sub-family of $\binom{[n']}{r'}$, where $n' = n+2$ and $r' = r+1$. We have $r' \leq n'/2$ as $r \leq n/2$. We now apply proper left-compressions to $\mathcal{J}$ until we obtain a compressed family $\mathcal{K}$. Since $[r] \in \mathcal{A} \cap \mathcal{B}$, we have $G := \{1\} \cup [3, r'+1]$ and $H := [2, r'+1]$ in $\mathcal{J}$. Note that $\{G, H\} \cup \mathcal{I}$ is a compressed family and hence it is contained in $\mathcal{K}$. Having determined that $[2, r'+1] \in \mathcal{K}$ and hence $\mathcal{K} = \mathcal{K}([2, r'+1])$ (since $\mathcal{K}$ is intersecting), we can now apply Theorem 5 to obtain $|\mathcal{K}| \leq |\mathcal{S}_{n',r'}([2, r'+1])| + 1 = \binom{n+1}{r} - \binom{n-r}{r} + 1$. Since $|\mathcal{K}| = |\mathcal{J}| = |\mathcal{G}| + |\mathcal{H}| + |\mathcal{I}| = |\mathcal{A}| + |\mathcal{B}| + \binom{n}{r-1}$, the result follows. □

Lemma 4 *Let* $2 \leq r \leq n/2$*, and let* $\mathcal{A}$ *be a non-centred intersecting sub-family of* $\binom{[n]}{r}$*. Let* $i, j \in [n], i \neq j$*, and suppose that* $\Delta_{i,j}(\mathcal{A})$ *is centred. Then* $|\mathcal{A}| \leq |\mathcal{N}|$.

Proof Having $\mathcal{A}$ non-centred and $\Delta_{i,j}(\mathcal{A})$ centred implies that $\mathcal{A} = \mathcal{A}(\{i,j\})$, $\mathcal{A}(i)(\bar{j}) \neq \emptyset$ and $\mathcal{A}(\bar{i})(j) \neq \emptyset$. So $|\mathcal{A}| = |\mathcal{A}(i)(j)| + |\mathcal{A}\langle i\rangle(\bar{j})| + |\mathcal{A}(\bar{i})\langle j\rangle|$ with $\mathcal{A}\langle i\rangle(\bar{j})$ and $\mathcal{A}(\bar{i})\langle j\rangle$ being non-empty sub-families of $\binom{[n]\backslash\{i,j\}}{r-1}$. Given that $\mathcal{A}$ is intersecting, $\mathcal{A}\langle i\rangle(\bar{j})$ and $\mathcal{A}(\bar{i})\langle j\rangle$ are cross-intersecting. By Theorem 6, $|\mathcal{A}\langle i\rangle(\bar{j})| + |\mathcal{A}(\bar{i})\langle j\rangle| \leq \binom{n-2}{r-1} - \binom{n-r-1}{r-1} + 1$. Since $|\mathcal{A}(i)(j)| \leq \binom{n-2}{r-2}$ and $|\mathcal{N}| = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$, the result follows. □

Proof of Theorem 2 We apply proper left-compressions to $\mathcal{A}$ until we obtain a family $\mathcal{A}^*$ such that $\mathcal{A}^*$ is invariant under any left-compression. If $\mathcal{A}^*$ is non-centred, then $[2, r+1] \in \mathcal{A}^*$ and hence the result follows by Theorem 5. If $\mathcal{A}^*$ is centred, then the result follows by Lemma 4. □

We now start working towards the proof of Theorem 4. We shall first develop some further notation.

Let $a := |W|$, and let $w_1, \ldots, w_a$ be the elements of W. For $i \in [p]$, let $V_i := X_i \backslash W$ and $b_i := |V_i|$, and let $v_{i1}, \ldots, v_{ib_i}$ be the elements of V_i; for the purpose of the left-compression operation, we put the elements of X_i in the order $w_1 < \cdots < w_a < v_{i1} < \cdots < v_{ib_i}$.

For $\mathcal{A} \subset \mathcal{U}$, let $\mathcal{A}_{(i)} := \{A \in \mathcal{A} \colon A \subset X_i\}$ and $\mathcal{A}_i := \mathcal{A}_{(i)}(V_i)$, $i = 1, \dots, p$.

We will use the following lemma when dealing with the extremal cases in Theorem 4; we will prove this lemma later.

Lemma 5 *Let $\mathcal{A}$ be as in Theorem 4. Suppose $r < \mu/2$ and, for some $i \in [p]$ and $x, y \in X_i$, $\Delta_{x,y}(\mathcal{A})$ is a star of $\mathcal{U}$ of size $|\mathcal{U}(w_1)|$. Then $\mathcal{A}$ is a star of $\mathcal{U}$ whose centre is in W.*

Proof of Theorem 4 We apply a composition of compressions $\Delta_{x,y}$, $x, y \in X_1$, $x < y$, to $\mathcal{A}$ until $\mathcal{A}_{(1)}$ is compressed. We then repeat this procedure for $\mathcal{A}_{(2)}, \dots, \mathcal{A}_{(p)}$ in the given order, and it is easy to see that after the i'th procedure we get that $\mathcal{A}_{(j)}$ is compressed for all $j \in [i]$. Clearly $\mathcal{A}$ remains intersecting and $\mathcal{A}_{(1)}, \dots, \mathcal{A}_{(p)}$ become compressed.

Clearly the families $\mathcal{A}_{(1)}, \mathcal{A}_2, \dots, \mathcal{A}_p$ partition $\mathcal{A}$. Let $\mathcal{J} := \mathcal{U}(w_1)$. Since $2 \le r \le \mu/2$, by taking X_1 and $X_1' := X_1 \backslash \{w_1\}$ to represent $[n]$ and Z respectively in Corollary 1, we get $|\mathcal{A}_{(1)}| = |\mathcal{A}_{(1)}(X_1')| \le |\mathcal{J}_{(1)}(X_1')| = |\mathcal{J}_{(1)}|$. Similarly, for $i = 2, \dots, p$, by taking X_i and V_i to represent $[n]$ and Z respectively in Corollary 1, we get $|\mathcal{A}_i| \le |\mathcal{J}_i|$. So $|\mathcal{A}| \le |\mathcal{J}|$ and hence (i).

Suppose $r < \mu/2$ and $\mathcal{A}_{(j)} \ne \mathcal{J}_{(j)}$ for some $j \in [p]$. Taking Z' to be X_1' or V_j, depending on whether $j = 1$ or $j > 1$ respectively, Corollary 1 gives us $|\mathcal{A}_{(j)}(Z')| < |\mathcal{J}_{(j)}(Z')|$. So $|\mathcal{A}| < |\mathcal{J}|$. Lemma 5 ensures that if $\mathcal{A}$ is initially non-centred, then the compressions mentioned above do not change $\mathcal{A}$ to $\mathcal{J}$. Hence (ii). □

We now come to the proof of Lemma 5, for which we need the lemma below that is often useful for determining the structure of extremal intersecting families.

Lemma 6 *Suppose $\emptyset \ne \mathcal{A} \subseteq \binom{X}{r}$, $2r < n := |X|$, such that, if $A \in \mathcal{A}$ and $B \in \binom{X \backslash A}{r}$, then $B \in \mathcal{A}$. Then $\mathcal{A} = \binom{X}{r}$.*

Proof Let $A \in \mathcal{A}$. Let B be an arbitrary set in $\binom{X}{r}$ that intersects A on exactly $r - 1$ elements. Since $n \ge 2r + 1$, we can choose $C \in \binom{X}{r}$ such that C is disjoint from $A \cup B$. By the assumption of the proposition, we have $C \in \mathcal{A}$, which in turn implies $B \in \mathcal{A}$. Repeated application of this step gives us that any set in $\binom{X}{r}$ is in $\mathcal{A}$. □

Lemma 7 *Let $\mathcal{A}$ be as in Theorem 4. Suppose $\mathcal{U}_{(j)}(x) \subseteq \mathcal{A}_{(j)}$ for some $x \in \bigcup_{i=1}^{p} X_i$ and $j \in [p]$. Then $\mathcal{A} \subseteq \mathcal{U}(x)$.*

Proof Since $\mathcal{U}_{(j)}(x) \subseteq \mathcal{A}_{(j)}$ and $|X_j| \ge \mu \ge 2r$, for all $B \in \mathcal{U} \backslash \mathcal{U}(x)$ we can find $A \in \mathcal{A}_{(j)}$ such that $A \cap B = \emptyset$. Since $\mathcal{A}$ is intersecting, the result follows. □

Proof of Lemma 5 Let c be the centre of $\Delta_{x,y}(\mathcal{A})$. So $\Delta_{x,y}(\mathcal{A}) = \mathcal{U}(c)$. Clearly the stars of $\mathcal{U}$ of largest size are those whose centres are in W. Thus, since $|\mathcal{U}(c)| = |\mathcal{U}(w_1)|$, $c \in W$. Let $\mathcal{J} := U(c)$ and $\mathcal{K} := \mathcal{U}(y)$.

Suppose $y \notin X_j$ for some $j \in [p]$. Then, since $\Delta_{x,y}(\mathcal{A}) = \mathcal{U}(c)$, we have $\mathcal{A}_{(j)} = \mathcal{J}_{(j)}$ and hence, by Lemma 7 and the fact that $|\mathcal{A}| = |\Delta_{x,y}(\mathcal{A})| = |\mathcal{J}|$, $\mathcal{A} = \mathcal{J}$. So we now assume that $y \in X_i$ for all $i \in [p]$, i.e. $y \in W$. If $\mathcal{A} = \mathcal{J}$ then we are done, so we assume that $\mathcal{A} \neq \mathcal{J}$. Together with $\Delta_{x,y}(\mathcal{A}) = \mathcal{J}$, this clearly implies that $c = x$. So $\Delta_{x,y}(\mathcal{A}) = \mathcal{J} = \mathcal{U}(x)$. Our next observation is that

$$A_1, A_2 \in \mathcal{K}\backslash\mathcal{J},\ A_1 \in \mathcal{A},\ A_1 \cap A_2 = \{y\} \quad \Rightarrow \quad A_2 \in \mathcal{A} \tag{4}$$

because otherwise, since $\Delta_{x,y}(\mathcal{A}) = \mathcal{J}$, we get $\delta_{x,y}(A_2) \in \mathcal{A}$ and $A_1 \cap \delta_{x,y}(A_2) = \emptyset$, which is a contradiction as $\mathcal{A}$ is intersecting.

If $A \in \mathcal{J} \cap \mathcal{K}$ then $A = \delta_{x,y}(A)$. So $\mathcal{J} \cap \mathcal{K} \subset \mathcal{A}$. Since $\mathcal{A} \neq \mathcal{J} = \Delta_{x,y}(\mathcal{A})$, there exists $B \in \mathcal{A}$ such that $\delta_{x,y}(B) \neq B$. So $B \in \mathcal{K}\backslash\mathcal{J}$. We have $B \subset X_j$ for some $j \in [p]$. Let $Y := X_j\backslash\{x\}$ and $\mathcal{Y} := \{A \in \binom{Y}{r} : y \in A\}$. Let $Z := Y\backslash\{y\}$ and $\mathcal{B} := \{A\backslash\{y\} \colon A \in \mathcal{A} \cap \mathcal{Y}\} \subseteq \binom{Z}{r-1}$. Since $B\backslash\{y\} \in \mathcal{B}$ and $\mathcal{A} \cap \mathcal{Y} \subseteq \mathcal{K}\backslash\mathcal{J}$, it follows by (4) and Lemma 6 that $\mathcal{B} = \binom{Z}{r-1}$. So $\mathcal{Y} \subset \mathcal{A}$ and hence, since we also have $\mathcal{J} \cap \mathcal{K} \subset \mathcal{A}$, $\mathcal{K}_{(j)} \subseteq \mathcal{A}_{(j)}$. By Lemma 7, $\mathcal{A} \subseteq \mathcal{K}$. Since $|\mathcal{K}| = |\mathcal{J}| = |\mathcal{A}|$, $\mathcal{A} = \mathcal{K}$. □

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